# $\mathbb{A}^1$- contractible varieties

references: • $\mathbb{A}^1$ - contractibility of affine modifications
- Dubouloz - P. - Østvær

• $\mathbb{A}^1$- homotopy theory and contractible varieties: a survey
- Asok - Østvær

base field $k$

$Sm_k$ = smooth varieties $/k$

$\mathcal{H}(k)$ = unstable $\mathbb{A}^1$- homotopy category$/k$

$SH(k)$ = stable — " —

Def: $X \in Sm_k$ is $\mathbb{A}^1$- contractible if $X \to \operatorname{Spec} k$ is an isomorphism in $\mathcal{H}(k)$.

$\mathbb{A}^1$- weak equivalences

Q: What are the isomorphisms in $\mathcal{H}(k)$?

Some examples:

- $Y \times \mathbb{A}^n \to Y$ $\qquad Y \in Sm_k$
- vector bundle maps
- $f: Y \to Z \in Sm_k$ that is "Nisnevich locally trivial"

  ie $\exists$ Nisnevich covering $u: U \to Z$ and an iso of $U$-schemes $Y \times_Z U \cong U \times_{\operatorname{Spec} k} W$

  sth $\mathbb{A}^1$-contr eg $\mathbb{A}^a$

## First examples of $\mathbb{A}^1$-contractible varieties

- $\mathbb{A}^n$
- quasi-affine variety (Winkelmann)

  $Q = \{x_1 x_2 - x_3 x_4 = x_5(1+x_5)\} \subseteq \mathbb{A}^5$

  $E = \{x_1 = x_3 = x_5 + 1 = 0\} \subseteq Q$

  $X := Q \setminus E$

Claim: $X$ is $\mathbb{A}^1$-contractible

Reason: $X$ is the quotient of a scheme theoretically free action

$$\mathbb{G}_a \curvearrowright \mathbb{A}^5$$

$\uparrow$
$\mathbb{A}^1$

$\mathbb{A}^5 \to X$ quotient map

- Asok–Doran: find more examples like of dim $\geq 4$

## Zariski Cancellation Problem

$$X \times \mathbb{A}^1 \cong \mathbb{A}^{\dim X + 1} \overset{?}{\Rightarrow} X \cong \mathbb{A}^{\dim X}$$

- yes if $\dim X \leq 2$
- no if $\dim X \geq 3$ and char $k > 0$
- unknown if $\dim X \geq 3$ and char $k = 0$

Note that $X \times \mathbb{A}^1 \cong \mathbb{A}^{\dim X + 1}$

$\Rightarrow$ $X$ $\mathbb{A}^1$-contractible + smooth + affine

From now on $k = \mathbb{C}$:

**Koras-Russell threefold**

$$KR = \{x^2y + t^3 + z^2 + x = 0\} \subseteq \mathbb{A}^4$$

$\underline{Q}$: Is $KR$ $\mathbb{A}^1$-contractible ?

Let's go back to classical topology and view $KR(\mathbb{C})$ as a smooth mfld.

Kaliman-Zaidenberg: $KR(\mathbb{C})$ is contractible using the fact that $KR$ is an affine modification of $\mathbb{A}^3$

Rmk: $\mathbb{A}^1$-contractible $\Rightarrow$ contractible

**Def (affine modification):**

Start with $X = \operatorname{Spec} A$ affine variety

$$(f, a_1, \dots, a_s) = I \subset A$$

$\uparrow$ regular sequence $s \geq 1$

$$Z = V(I)$$

$$D = \operatorname{div} f$$

$$
\begin{array}{ccccc}
E & \subseteq & Bl_Z X & \subseteq & X \times \mathbb{P}^s \\
\cup\!\! | & & \cup\!\! | & & \cup\!\! | \\
\tilde{E} & \subseteq & \tilde{X} & \subseteq & X \times \mathbb{A}^s
\end{array}
$$

$\leftarrow y_0, \ldots, y_s$

$= \{y_0 \neq 0\}$

exc div of affine mod

affine modification

<u>Ex:</u> $X = \mathbb{A}^3$

$I = (x^2, -x - t^3 - z^2)$

$f = x^2$

$$Bl_Z X = \{ y_1 x^2 + y_0 (x + t^3 + z^2) = 0 \} \subseteq \mathbb{A}^3 \times \mathbb{P}^1$$

$$\tilde{X} = \{ y x^2 + x + t^3 + z^2 = 0 \} \subseteq \mathbb{A}^3 \times \mathbb{A}^1$$

$\tilde{X} = KR$

$$
\begin{tikzcd}
Z \times \mathbb{A}^s \cong \tilde{E} \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d] & \tilde{X} - \tilde{E} \arrow[l, hook'] \arrow[d, "\cong"] \\
Z \arrow[d, hook] & & \\
D \arrow[r, hook] & X & X - D \arrow[l, hook']
\end{tikzcd}
$$

$\leftarrow$ restriction of blow up map

Assume $Z \hookrightarrow D$ induces an iso in homology (+ some more technical assumption)

5-lemma $\Rightarrow$ $\tilde{X}$ and $X$ have same homology

Kaliman-Zaidenberg find criteria for when $X$ and $\tilde{X}$ also have the same fundamental group.

So assume that $X$ (resp $\tilde{X}$) is contractible $\Rightarrow$ all homology and fundamental group of $X$ (resp $\tilde{X}$) are trivial

$\Rightarrow$ all homology and fundamental group of $\tilde{X}$ (resp $X$) are trivial

assume everything above is satisfied

Hurewicz $\Rightarrow$ all homotopy

groups of $\tilde{X}$ (resp $X$)
are trivial

Whitehead $\Rightarrow$ $\tilde{X}$ (resp $X$) is contractible

## $\mathbb{A}^1$-contractibility of affine modifications

<u>Thm</u> (Dubouloz - P. - Østvær)

$X = \operatorname{Spec} A$, $D = \operatorname{div} f$, $Z = V(I)$ as above but all smooth

and st :

- $Z \hookrightarrow D$ $\mathbb{A}^1$-weak eq
- supports of $D$ and $\tilde{E}$ are irred
- $\tilde{X}$ is $\mathbb{A}^1$-contractible

$\Rightarrow$ $X$ is $\mathbb{A}^1$-contractible

Pf :

$$\begin{array}{ccccc}
\tilde{X} - \tilde{E} & \longrightarrow & \tilde{X} & \longrightarrow & \tilde{X}/\tilde{X}-\tilde{E} \\
\downarrow \cong & & \downarrow & & \downarrow \simeq_{\mathbb{A}^1} \\
X - D & \longrightarrow & X & \longrightarrow & X/X-D
\end{array}$$

Claim: $\tilde{X}/\tilde{X}-\tilde{E} \to X/X-D$ is an $\mathbb{A}^1$-weak eq

Reason: Purity: $\tilde{X}/\tilde{X}-\tilde{E} \simeq_{\mathbb{A}^1} Th(\underbrace{N_{\tilde{E}}\tilde{X}}_{\tilde{E}\times\mathbb{A}^1})$

$$\tilde{E} \cong Z \times \mathbb{A}^s$$

$$\begin{array}{c} \wr\! 2 \\ \tilde{E}_+ \wedge \mathbb{P}^1 \\ \downarrow \simeq_{\mathbb{A}^1} \\ Z_+ \wedge \mathbb{P}^1 \\ \downarrow \simeq_{\mathbb{A}^1} \end{array}$$

extra base pt

$$X/X-D \simeq D_+ \wedge \mathbb{P}^1$$

$$\begin{tikzcd}
{[\tilde{X}-\tilde{E}, Y]} & {[\tilde{X}, Y]} \arrow[l] \arrow[r, phantom, "\cong *" above] & {[\tilde{X}/\tilde{X}-\tilde{E}, Y]} \arrow[l] & {[\Sigma \tilde{X}-\tilde{E}, Y]} \arrow[l] \\
{[X-D, Y]} \arrow[u, "\cong"] & {[X, Y]} \arrow[l] \arrow[u] & {[X/X-D, Y]} \arrow[l] \arrow[u, "\cong"] & {[\Sigma X-D, Y]} \arrow[l] \arrow[u, "\cong"]
\end{tikzcd}$$

Want to show that $X$ is $\mathbb{A}^1$-contr

$\iff$ $[X, Y] \cong * \quad \forall\, Y \in \mathcal{H}(\mathbb{C})$

maps in $\mathcal{H}(\mathbb{C})$

half of S-lemma

$\Rightarrow$ $[X,Y] \to [\tilde{X}, Y]$ is injective

$\Rightarrow$ $[X,Y] = *$

$\Rightarrow$ $X$ is $\mathbb{A}^1$-contr. $\square$

<u>Ex</u>: $X = KR = \{x^2y + x + t^3 + z^2 = 0\} \subseteq \mathbb{A}^4$

$f = x \quad I = (x, t+1, z-1)$

$\Rightarrow$ affine modification $\tilde{X} = \mathbb{A}^3$

⚠ This does not prove that $KR$ is $\mathbb{A}^1$-contractible because of smoothness assumption

Hoyois - Krishna - Østvær: $KR$ is <u>stably</u> $\mathbb{A}^1$-contractible ie

$\Sigma^\infty_{\mathbb{P}^1} KR \to \Sigma^\infty_{\mathbb{P}^1} \operatorname{Spec} \mathbb{C}$ is an iso in $SH(\mathbb{C})$

$\Leftrightarrow$ $\exists n$ st $KR \wedge (\mathbb{P}^1)^{\wedge n}$ is

## $\mathbb{A}^1$ - contractible

They show that if $X \times Y \to Y$ induces an iso in motivic cohomology (= higher Chow groups)

$\forall$ smooth affine variety $Y$

$\Rightarrow$ $X$ is stably $\mathbb{A}^1$- contractible

## Dubouloz - Fasel: $KR$ is $\mathbb{A}^1$- contractible

$$\begin{tikzcd}
\mathbb{A}^2 - 0 \arrow[r, hook] \arrow[d, "\simeq_{\mathbb{A}^1}"', "\leftarrow \text{hard part}"] & \mathbb{A}^2 = \{y=0\} \arrow[r] \arrow[d] & \mathbb{A}^2/\mathbb{A}^2 - 0 \simeq_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2} \arrow[d, "\simeq_{\mathbb{A}^1}"] \\
KR \setminus L \arrow[r, hook] & KR \arrow[r] & KR/KR - L \simeq L_+ \wedge (\mathbb{P}^1)^{\wedge 2}
\end{tikzcd}$$

$L = \{x = t = z = 0\}$

$KR = \{x^2 y + x + t^3 + z^2 = 0\} \subseteq \mathbb{A}^4$

same half 5-lemma argument as above shows that $KR$ is $\mathbb{A}^1$- contractible

There is only 1 smooth $\mathbb{A}^1$-contractible variety of dim 1, namely $\mathbb{A}^1$.

Q: Are there smooth $\mathbb{A}^1$-contractible varieties of dim 2 $\not\cong \mathbb{A}^2$?

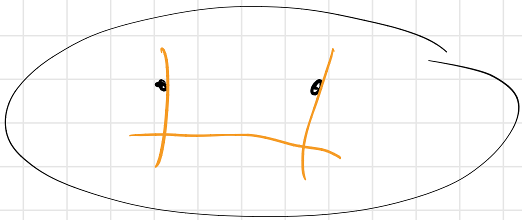
Ex: tom Dieck - Petrie surface

$$\left\{ \frac{(xz+1)^2 - (yz+1)^3}{z} = 1 \right\} \subseteq \mathbb{A}^3$$

smooth surface
+ contractible
since affine
modification of
$\mathbb{A}^2$ (Kaliman
- Zaidenberg)

Hoyois - Krishna - Østvær
$\Rightarrow$ this is stably
$\mathbb{A}^1$-contractible

**Q:** $\mathbb{A}^1$-contractibility $\overset{?}{\Rightarrow}$ $\mathbb{A}^1$-chain-connectedness / naive $\mathbb{A}^1$-connectedness



If yes then $\mathbb{A}^1$-contractibility would imply $\bar{\kappa} = -\infty$ $\Rightarrow$ $\mathbb{A}^2$ would be the only smooth $\mathbb{A}^1$-contractible variety of dim 2.